Bridgeland stability conditions on some Picard rank one varieties

Stavanger workshop

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Review: Bridgeland Stability Condition

Stability conditions: $\sigma = (Z, \mathcal{P})$ defined on a $\mathbb{C}$-linear triangulated category $\mathcal{T}$.

- The kernel of the central charge Z avoids all 'stable characters'
- Support condition: the 'cone neighborhood' of ker$Z$ avoids all 'stable characters'
- $\mathcal{T}$: $D^b_{coh}(X)$
  X: Picard rank one (smooth projective connected) varieties over $\mathbb{C}$

All examples are of dimension two or three, but statements may also hold for higher dimensional varieties.

Review: Geometric stability conditions

Let X be a smooth projective connected surface.

- Geometric stability condition: all skyscraper sheaves are stable with the same phase.
- Non-degenerate: the image of the central charge is not contained in a real line.

When X is of Picard rank 1 with ample generator H, ker$Z$ in $K_{\mathbb{R}}(X)$ is a line: a point in $P(K_{\mathbb{R}}(X))$.

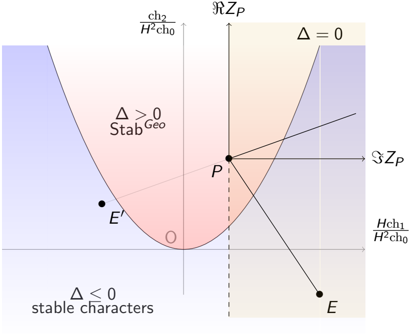
- Bound for both slope stable and σ-stable: Bogolomov inequality

$$\Delta_H = (H\mathrm{ch}_1)^2 - 2\mathrm{ch}_2(H^2\mathrm{ch}_0) \geq 0$$

- Point (avoids stable characters) on $P(K_{\mathbb{R}}(X))$
  $\leftrightsquigarrow$ Kernel of the Central charge

The CCC-plane

We use Chern characters $\{H^2\mathrm{ch}_0, H\mathrm{ch}_1, \mathrm{ch}_2\}$ for the bases of the projective plane $P(K_{\mathbb{R}}(X))$, and call it the CCC-plane.
The line at infinity: $\mathrm{ch}_0 = 0$.



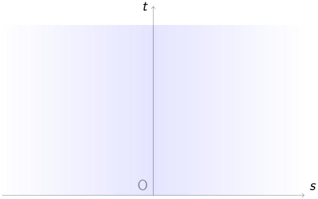
Let $(1, s, q)$ be a point on the CCC-plane, then $\sigma_{s,q} = (Z_{s,q}, \mathcal{P}_{s,q})$ is constructed as follows:

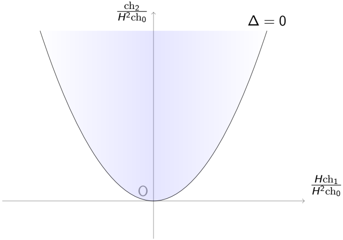
- $\mathcal{P}_{s,q}((0,1]) := \mathrm{Coh}_s(X)$.
- $Z_{s,q} = -(\mathrm{ch}_2 - q \cdot \mathrm{ch}_0) + \mathrm{i}(\mathrm{ch}_1 - s \cdot \mathrm{ch}_0)$.

When a neighborhood of $(1, s, q)$ is not below any stable character on the CCC-plane, $\sigma_{s,q}$ is a geometric stability condition.

The CCC-plane/ the upper half plane

- Relation with the upper half plane model: $\sigma'_{s,t} \leftrightsquigarrow \sigma_{s,\frac{s^2}{2}+\frac{t^2}{2}}$

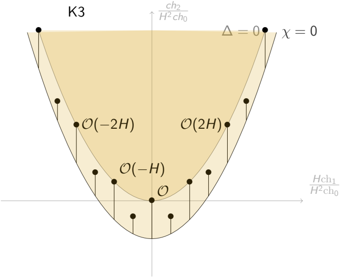


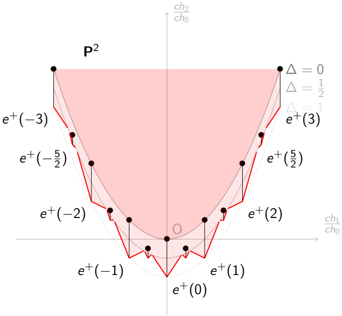


- Upper half plane model: $\sigma'_{s,t} = (Z'_{s,t}, \mathcal{P}'_{s,t})$
- $Z'_{s,t} = -\mathrm{ch}_2^s + \frac{t^2}{2}\mathrm{ch}_0 + \mathrm{i}t\mathrm{ch}_1^s$

Examples: space of the geometric stability conditions

In the Picard rank one case, $\mathrm{Stab}^{Geo}/\widetilde{GL(2,\mathbb{R})} \hookrightarrow$ CCC-plane.

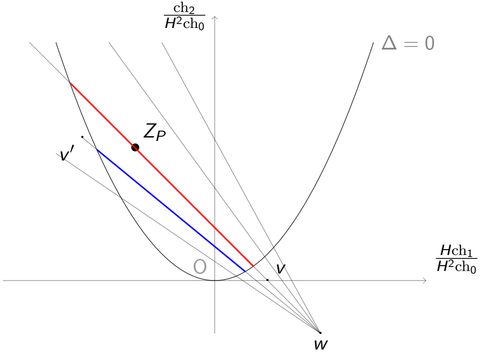




Potential walls

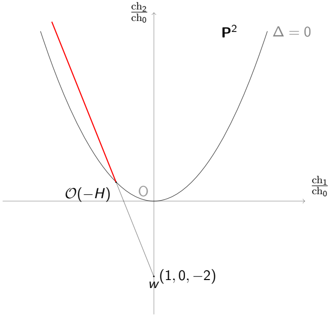
Given two Chern characters w and v, suppose their projections on the CCC-plane are well-defined, the potential wall of them is the straight line across them. The 'nested wall' property is clear.

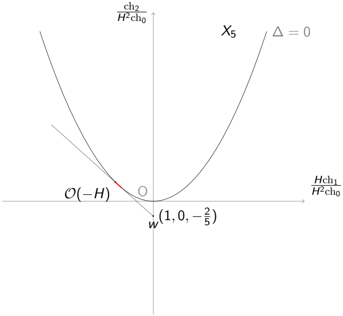
- $Z(w)//Z(v)$
- $\ker Z$, w and v are colinear



Examples: potential walls

We consider the first wall of $\mathrm{Hilb}^2(X)$ for two different X.



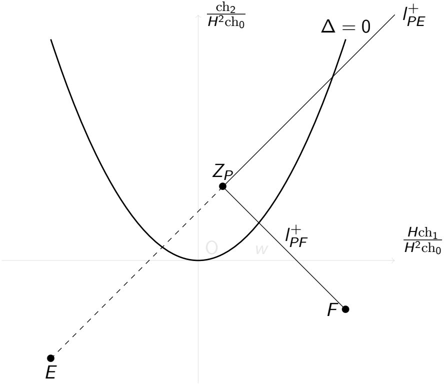


Compare the slope

Let $P = (1, s, q)$ be a point on the CCC-plane, E and F be two objects in Coh_s. The inequality

$$\sigma_P(E) > \sigma_P(F)$$

holds if and only if the ray l^+_{PE} is above l^+_{PF}.



CCC-plane: From surfaces to higher dimensional varieties

When X is a higher dimensional variety with Picard rank one, the same story works for the first tilt slope function.

- H: Ample divisor.
- The first tilt heart $\mathrm{Coh}_{sH}(X) := \langle \mathcal{T}_{sH}(X), \mathcal{F}_{sH}(X)[1] \rangle$
- 'Reduced' central charge $t > 0$:

$$\overline{Z}_{s,t}(E) = 3tH^2\mathrm{ch}_1^{sH}(E) + i\left(\sqrt{3}H\,\mathrm{ch}_2^{sH}(E) - \frac{\sqrt{3}}{2}t^2H^2\mathrm{ch}_0^{sH}(E)\right).$$

- $\nu_{s,t}$: tilt slope function $\frac{\Im Z_{s,t}}{\Re Z_{s,t}}$
- On the CCC-plane, the kernel of $\overline{Z}_{s,t}$ is $(1, s, \frac{t^2+s^2}{2})$.

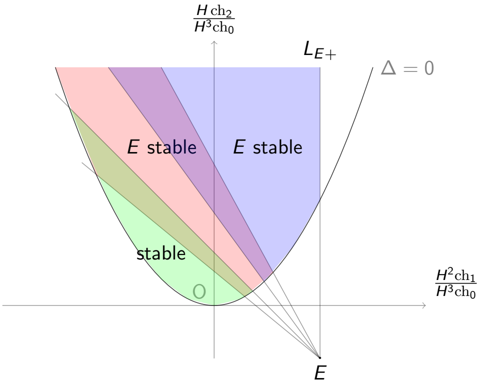
Stability condition on three-fold (Review of the progress)

The existence of Bridgeland stability conditions on 3-fold.

- Bayer, Bertram, Macrì, Toda: the 'double tilting'
- Conjectural Bogolomov-Gieseker type inequality involves the third Chern character
- $\mathbf{P}^3$: Macrì; Quadratic hypersurface: Schmidt
- Abelian 3-fold (Picard rank one): Maciocia, Piyaratne
- Finite quotient of Abelian 3-fold: Bayer, Macrì, Stellari.

Stability condition on three-fold: reduce to small t

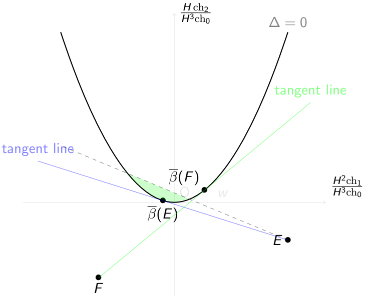
- Stability conditions on abelian threefolds and some Calabi-Yau threefolds. Bayer, Macrì, Stellari.
- Picard rank one case: Conjecture 5.3



Conjecture 5.3: Definition of $\overline{\beta}(E)$

Let E be an object in $D^b(X)$ such that $\mathrm{ch}_0(E)$ and $\mathrm{ch}_1(E)$ are not both 0.

$$\overline{\beta}(E) = \begin{cases} \dfrac{H\,\mathrm{ch}_2(E)}{H^2\mathrm{ch}_1(E)} & \text{if } \mathrm{ch}_0(E) = 0, \\ \dfrac{H^2\mathrm{ch}_1(E) - \sqrt{\overline{\Delta}_H(E)}}{H^3\mathrm{ch}_0(E)} & \text{if } \mathrm{ch}_0(E) \neq 0, \end{cases}$$



We call an object E: $\overline{\beta}$*-stable* if there exists an open neighbor hood $U \subset \mathbb{R}^2$ of $(0, \overline{\beta}(E))$ such that for any $(t, s) \in U$ with $t > 0$, either E or $E[1]$ is a $\nu_{s,t}$ tilt-stable object of $\mathrm{Coh}_{\beta}(E)$.

The Conjecture

Conjecture (Bayer, Macrì, Stellari)

Let $E \in D^b(X)$ be a $\overline{\beta}$-stable object, then

$$\mathrm{ch}_3^{\overline{\beta}(E)H}(E) \leq 0.$$

When X is of Picard rank one, the conjecture implies the existence of stability conditions on $D^b(X)$.

Picard rank one Fano 3-fold

Theorem

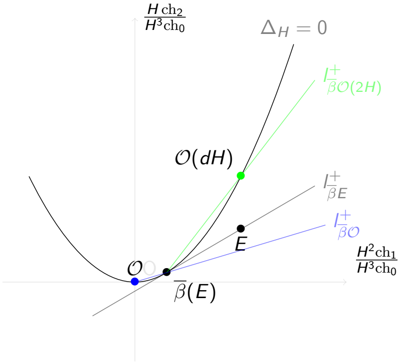
The conjecture holds for Picard rank one Fano 3-folds.

Classification of Picard rank one Fano 3-folds:

- Index: 1,2,3,4
- Degree: up to 22

Idea of the proof: compare the slope

X: Picard rank one Fano 3-fold. H: the ample divisor generator.
d: the index, $K_X = -dH$.
May assume $0 \leq \overline{\beta}(E) < 1$.



- $\mathrm{Hom}(\mathcal{O}(dH), E) = 0$
- $\mathrm{Hom}(E, \mathcal{O}[1]) = 0$
- $\mathrm{Hom}(\mathcal{O}(dH), E[2]) = 0$

$$\chi(\mathcal{O}(sH), E) \leq 0,$$

for $1 \leq s \leq d$.

Fano 3-folds: HRR formula

Hirzebruch-Riemann-Roch:

$$\chi(E) = \mathrm{ch}_3(E) + \mathrm{td}_1(X)\mathrm{ch}_2(E) + \mathrm{td}_2(X)\mathrm{ch}_1(E) + \mathrm{td}_3(X)\mathrm{ch}_0(E).$$

- X Fano: $\mathrm{td}_i(X)$ is 'positive'.
- Expand $\chi(\mathcal{O}(sH), E)$ in terms of $\mathrm{ch}_i^{\overline{\beta}}(E)$, the coefficient for $\mathrm{ch}_1^{\overline{\beta}}(E)$ is 'positive'.

Example

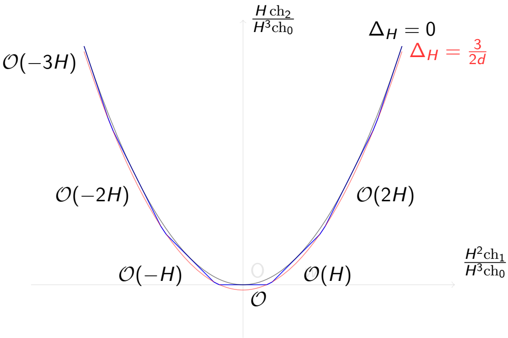
X: smooth cubic 3-fold in $\mathbf{P}^4$.Hirzebruch-Riemann-Roch:

$$\chi(E) = \mathrm{ch}_3(E) + H\mathrm{ch}_2(E) + \frac{2}{3}H^2\mathrm{ch}_1(E) + \frac{1}{3}H^3\mathrm{ch}_0(E).$$

- $0 \geq \chi(\mathcal{O}(H), E)$, $\chi(\mathcal{O}(2H), E)$.
- $\mathrm{ch}_3^{\overline{\beta}}(E) + \overline{\beta}\mathrm{ch}_2^{\overline{\beta}}(E) + \left(\frac{\overline{\beta}^2}{2} + \frac{1}{6}\right) H^2\mathrm{ch}_1^{\overline{\beta}}(E) + \left(\frac{\overline{\beta}^3}{6} + \frac{\overline{\beta}}{6}\right) H^3\mathrm{ch}_0^{\overline{\beta}}(E)$
- $\mathrm{ch}_3^{\overline{\beta}}(E) + (\frac{\overline{\beta}}{2} - \overline{\beta} + \frac{2}{3})H^2\mathrm{ch}_1^{\overline{\beta}}(E) + (\frac{\overline{\beta}^3}{6} - \frac{\overline{\beta}^2}{2} + \frac{2\overline{\beta}}{3} - \frac{1}{3})H^3\mathrm{ch}_0^{\overline{\beta}}(E)$

Index one case

The naive idea fails.
A stricter bound for stable characters:



- $\chi(E,E) \leq 1$. Above the curve $\Delta_H = \frac{3}{2d}$ implies the '$=$'.
- Above the tangent lines implies $\mathrm{Ext}^2(E,E) = 0$.

Remark

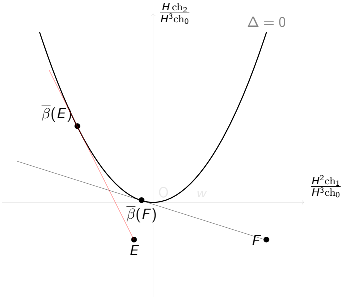
- Higher Picard rank: partial result.
  - $\mathrm{Bl}_{pt}\mathbf{P}^3$: ω and B are parallel to K_X, or $2H_0 - H_1$.
- Calabi-Yau 3-fold: ??

Suppose E and F are both $\overline{\beta}_H$-stable such that

$$\overline{\beta}_H(E) < \overline{\beta}_H(F),$$

then

$$\mathrm{Hom}(F, E) = 0.$$



Thank you!